

## COUNTEREXAMPLES ON SPECTRA OF SIGN PATTERNS

YAROSLAV SHITOV

ABSTRACT. An  $n \times n$  *sign pattern*  $S$ , which is a matrix with entries  $0, +, -$ , is called *spectrally arbitrary* if any monic real polynomial of degree  $n$  can be realized as a characteristic polynomial of a matrix obtained by replacing the non-zero elements of  $S$  by numbers of the corresponding signs. A sign pattern  $S$  is said to be a *superpattern* of those matrices that can be obtained from  $S$  by replacing some of the non-zero entries by zeros. We develop a new technique that allows us to prove spectral arbitrariness of sign patterns for which the previously known *Nilpotent Jacobian* method does not work. Our approach leads us to solutions of numerous open problems known in the literature. In particular, we provide an example of a sign pattern  $S$  and its superpattern  $S'$  such that  $S$  is spectrally arbitrary but  $S'$  is not, disproving a conjecture proposed in 2000 by Drew, Johnson, Olesky, and van den Driessche.

## 1. CONJECTURES

The study of spectra of matrix patterns deserved a significant amount of attention in recent publications. The conjecture mentioned in the abstract appeared in one of the foundational papers on this topic ([10]), and many subsequent works proved it in different special cases ([3, 7, 12, 14, 15]). One of the known sufficient conditions for superpatterns to be spectrally arbitrary is the *Nilpotent Jacobian* condition ([2, 10]), which allowed to solve several intriguing problems on this topic ([4, 11, 19]). Despite these efforts, the *superpattern conjecture* remained open by now, and we mention [5, 17, 18] as other recent work disussing this conjecture.

**Conjecture 1.** (Conjecture 16 in [10].) *If  $S$  is a minimal spectrally arbitrary sign pattern, then any superpattern of  $S$  is spectrally arbitrary.*

We note that this conjecture involves the concept of a *minimal* spectrally arbitrary sign pattern, that is, a sign pattern  $S$  which is spectrally arbitrary but is not a superpattern of any other spectrally arbitrary sign pattern. In our paper, we construct a sign pattern  $S$  and its superpattern  $S'$  such that  $S$  is spectrally arbitrary but  $S'$  is not. We do not investigate the question of minimality of  $S$ , but  $S$  is anyway a superpattern of some minimal spectrally arbitrary pattern  $S_0$ , and the pair  $(S_0, S')$  provides a counterexample to Conjecture 1 even if  $S$  is not minimal.

As said above, the *Nilpotent Jacobian* condition is sufficient for a zero pattern (and every superpattern of it) to be spectrally arbitrary. Our results show that this condition is not necessary, answering the questions posed explicitly in [1, 10, 17].

As a byproduct of our approach, we obtain solutions of two other related problems on the topic. Namely, we construct a sign pattern  $U$  such that  $\text{diag}(U, U)$  is

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spectrally arbitrary but  $U$  itself is not. This gives a solution to the problem posed in Section 5 in [8] and an answer to Question 3 in [5].

An  $n \times n$  sign pattern  $S$  is said to allow *arbitrary refined inertias* if, for any family  $n_+, n_-, n_0, n_i$  of nonnegative integers such that  $n_+ + n_- + n_0 + 2n_i = n$ , there is a matrix with sign pattern  $S$  which has  $n_+$  eigenvalues with positive real part,  $n_-$  eigenvalues with negative real part,  $n_0$  zero eigenvalues, and  $n_i$  purely imaginary eigenvalues. We provide an example of a sign pattern that allows arbitrary refined inertias but is not spectrally arbitrary, which solves the problem asked in Section 5 in [8] and in Section 5 in [13].

## 2. COUNTEREXAMPLES

We define the sign patterns

$$T = \begin{pmatrix} + & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & - & 0 \end{pmatrix}, \quad T' = \begin{pmatrix} + & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 \\ + & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & - & 0 \end{pmatrix},$$

which agree at every entry except  $(3, 1)$ , so  $T'$  is indeed a superpattern of  $T$ . Also, we fix any  $2 \times 2$  spectrally arbitrary pattern<sup>1</sup> and denote it by  $D$ . Let us prove several observations, which we will put together in the theorem below.

*Observation 2.* Let  $R$  be a matrix obtained from  $T'$  by replacing the signs with non-zero real numbers. Then  $R$  is not nilpotent.

*Proof.* The coefficients of  $t^3$  and  $t^5$  in the characteristic polynomial of  $R$  are equal to  $-r_{12}r_{23}r_{31} + (r_{11} + r_{22})r_{56}r_{65}$  and  $-r_{11} - r_{22}$ , respectively. These coefficients can vanish simultaneously only if  $r_{12}r_{23}r_{31} = 0$ .  $\square$

*Observation 3.* Let  $R$  be a matrix obtained from  $T$  by replacing the signs with non-zero real numbers. Assume that  $t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$  is the characteristic polynomial of  $R$ . Then  $a_3 = 0$  if and only if  $a_5 = 0$ .

*Proof.* The coefficients of  $t^3$  and  $t^5$  in the characteristic polynomial of  $R$  are equal to  $(r_{11} + r_{22})r_{56}r_{65}$  and  $-r_{11} - r_{22}$ , respectively. Therefore, the former of them is zero if and only if the latter is zero.  $\square$

*Observation 4.* The sign pattern  $\text{diag}(T, D)$  is not spectrally arbitrary.

*Proof.* If  $f = (t^2 + t + 1)(t^2 - t + 2)(t^2 + 1)(t^2 - 1)$  is realizable as the characteristic polynomial of a matrix with sign pattern  $\text{diag}(T, D)$ , then  $f$  has a divisor realizable as the characteristic polynomial of a matrix with sign pattern  $T$ . A straightforward checking of possible cases leads to a contradiction with Observation 3.  $\square$

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<sup>1</sup>In fact, spectrally arbitrary  $n \times n$  sign patterns exist for all  $n \geq 2$ , see [16].

In order to proceed, we consider the matrix

$$X = \begin{pmatrix} x_1 & 1 & 0 & 0 & 0 & 0 \\ -x_4 & -x_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -x_6 & -x_5 & 0 & 0 & 0 & 1 \\ x_7 & x_8 & x_9 & 0 & -x_3 & 0 \end{pmatrix},$$

whose sign pattern is  $T$  whenever the  $x_i$ 's take positive values.

*Observation 5.* For all  $b, c, d$ , there are positive values of the  $x_i$ 's such that the characteristic polynomial of  $X$  equals  $(t^2 + b)(t^2 + c)(t^2 + d)$ .

*Proof.* First, we assume that  $x_1, x_3, x_8, x_9$  are arbitrary and check that the matrix  $X$  defined by  $x_2 = x_1$ ,  $x_4 = b + c + d + x_1^2 - x_3$ ,  $x_5 = bc + bd + cd - bx_3 - cx_3 - dx_3 + x_3^2 + x_9$ ,  $x_6 = bcx_1 + bdx_1 + cdx_1 - bx_1x_3 - cx_1x_3 - dx_1x_3 + x_1x_3^2 + x_8 + x_1x_9$ ,  $x_7 = -bcd + x_1x_8 - bx_9 - cx_9 - dx_9 + x_3x_9$  has a desired characteristic polynomial. Picking  $x_3 = 1$  and defining  $x_1$  as a large enough positive number, we make  $x_2, x_3, x_4$  positive regardless of the values of  $x_8, x_9$ . Finally, the choice of  $x_9$  allows us to make  $x_5$  positive, and now  $x_6, x_7$  tend to  $+\infty$  as  $x_8$  gets large.  $\square$

*Observation 6.* If  $a_3/a_5 > 0$ , then there are positive values of the  $x_i$ 's such that the characteristic polynomial of  $X$  equals  $t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ .

*Proof.* Again, we assume that  $x_1, x_8, x_9$  are arbitrary and check that the matrix  $X$  defined by  $x_2 = a_5 + x_1$ ,  $x_3 = a_3/a_5$ ,  $x_4 = (-a_3 + a_4a_5 + a_5^2x_1 + a_5x_1^2)/a_5$ ,  $x_5 = (a_3^2 - a_3a_4a_5 + a_2a_5^2 + a_5^2x_9)/a_5^2$ ,  $x_6 = (a_1a_5^2 + a_3^2x_1 - a_3a_4a_5x_1 + a_2a_5^2x_1 + a_5^2x_8 + a_5^3x_9 + a_5^2x_1x_9)/a_5^2$ ,  $x_7 = (-a_0a_5 + a_5x_1x_8 + a_3x_9 - a_4a_5x_9)/a_5$  has a desired characteristic polynomial. Defining  $x_1$  as a large enough positive number, we make  $x_2, x_3, x_4$  positive regardless of the values of  $x_8, x_9$ . As in the proof of the previous observation, the choice of  $x_9$  allows us to make  $x_5$  positive, and then  $x_6, x_7$  tend to  $+\infty$  as  $x_8$  gets large.  $\square$

*Observation 7.* Let  $f$  be a monic real polynomial of degree 16. Then  $f$  has a divisor realizable as the characteristic polynomial of a matrix with sign pattern  $T$ .

*Proof.* Clearly,  $f$  is the product of eight quadratics of the form  $t^2 + a_it + b_i$ . If  $b_i$  is negative, then such a quadratic has two roots of different signs, and this allows us to assume that at least seven of the initial quadratics have their  $b_i$ 's nonnegative. By the pigeonhole principle, among these seven quadratics there are three that either have all  $a_i$ 's positive, or all  $a_i$ 's negative, or all  $a_i$ 's zero. In the first two cases, the product of these three quadratics is a polynomial as in Observation 6, and the case of zero  $a_i$ 's corresponds to Observation 5.  $\square$

*Observation 8.* Let  $V = \text{diag}(T, \dots, T, D, \dots, D)$  be a sign pattern of size  $(6t + 2d)$ . ( $T$  occurs  $t$  times,  $D$  occurs  $d$  times.) If  $d \geq 5$ , then  $V$  is spectrally arbitrary.

*Proof.* The result is true for  $t = 0$  because  $D$  is spectrally arbitrary (see also Proposition 2.1 in [9]). Now let  $t > 0$  and let  $f$  be a monic real polynomial of degree  $6t + 2d$  (which is at least 16). We apply Observation 7 and find a polynomial  $h$  that divides  $f$  and arises the characteristic polynomial of a matrix  $M_1$  with sign pattern  $T$ . Using the inductive assumption, we find a matrix  $M_2$  with characteristic polynomial  $f/h$  and sign pattern that has the same form as  $V$  but with one  $T$ -block

removed. Now the matrix  $\text{diag}(M_1, M_2)$  has sign pattern  $V$  and characteristic polynomial  $f$ .  $\square$

*Observation 9.* For any family  $\nu = (n_+, n_-, n_0, n_i)$  of nonnegative integers such that  $n_+ + n_- + n_0 + 2n_i = 8$ , there is a family  $\mu \leq \nu$  and a matrix  $M$  with sign pattern  $T$  and refined inertia  $\mu$ .

*Proof.* If  $n_0 + 2n_i \geq 6$ , then we are done because of Observation 5. Otherwise, we have  $n_+ + n_- \geq 3$ , and it suffices to check that any tuple  $\mu = (m_+, m_-, m_0, m_i)$  with  $m_+ + m_- \geq 3$  arises as a refined inertia of a matrix with sign pattern  $T$ .

Now we see that one of the tuples  $\mu - (3, 0, 0, 0)$ ,  $\mu - (0, 3, 0, 0)$ ,  $\mu - (2, 1, 0, 0)$ ,  $\mu - (1, 2, 0, 0)$  consists of nonnegative integers, and this tuple corresponds to some monic polynomial  $h$  of degree 3. We note that, for a sufficiently large positive  $N$ , the polynomials  $(t - N)^3h$ ,  $(t + N)^3h$ ,  $(t + 3N)(t - N)^2h$ ,  $(t - 3N)(t + N)^2h$  satisfy the condition as in Observation 6. As said above, one of these polynomials has  $\mu$  as a refined inertia.  $\square$

*Observation 10.* The sign pattern  $\text{diag}(T, D)$  allows arbitrary refined inertias.

*Proof.* Let  $\nu = (n_+, n_-, n_0, n_i)$  be a family of nonnegative integers such that  $n_+ + n_- + n_0 + 2n_i = 8$ . By Observation 9, there is a family  $\mu \leq \nu$  and a matrix  $M_1$  with sign pattern  $T$  and refined inertia  $\mu$ . Since  $D$  is spectrally arbitrary, it allows a matrix  $M_2$  with refined inertia  $\nu - \mu$ , and then the matrix  $\text{diag}(M_1, M_2)$  has sign pattern  $\text{diag}(T, D)$  and refined inertia  $\nu$ .  $\square$

Now we put all the observations together and conclude the paper.

**Theorem 11.** *Let  $T, T', D$  be as above. Then*

- (1) *the sign pattern  $S = \text{diag}(T, D, D, D, D, D)$  is spectrally arbitrary, but its superpattern  $S' = \text{diag}(T', D, D, D, D, D)$  is not spectrally arbitrary;*
- (2)  *$\text{diag}(T, D)$  allows arbitrary refined inertias but is not spectrally arbitrary;*
- (3) *there is a sign pattern  $U$  such that  $\text{diag}(U, U)$  is spectrally arbitrary but  $U$  is not.*

*Proof.* The definition of  $T$  and  $T'$  immediately shows that  $S'$  is a superpattern of  $S$ . By Observation 2,  $S'$  does not allow a nilpotent matrix, so it is not spectrally arbitrary. Observation 8 shows that  $S$  is spectrally arbitrary and completes the proof of (1).

The sign pattern  $\text{diag}(T, D)$  is not spectrally arbitrary by Observation 4, and it allows arbitrary refined inertias by Observation 10. This proves (2).

Finally, let  $U_1 = \text{diag}(T, D)$ . If  $U_2 = \text{diag}(U_1, U_1)$  is spectrally arbitrary, then the proof of (3) is complete. Otherwise, we define  $U_3 = \text{diag}(U_2, U_2)$ , and we are done if  $U_3$  is spectrally arbitrary. If this is still not the case, we complete the proof because  $\text{diag}(U_3, U_3)$  is spectrally arbitrary by Observation 8.  $\square$

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NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 20 MYASNITSKAYA ULITSA,  
MOSCOW 101000, RUSSIA

*E-mail address:* yaroslav-shitov@yandex.ru